HALL ALGEBRAS FOR ODD PERIODIC TRIANGULATED CATEGORIES

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ABSTRACT. We define the Hall algebra associated to any triangulated category under some finiteness conditions with the t-periodic translation functor T for odd $t > 1$. This generalizes the results in [20] and [23].

1. INTRODUCTION

Hall algebras provided a framework involving the categorification and the geometrization of Lie algebras and quantum groups in the past two decades (see [11, 12, 14, 15, 22]). In a broad sense, a Hall algebra provides a tool allowing one to code a category.

Let $\mathcal{A}$ be a finitary category, i.e., a (small) abelian category satisfying: (1) $|\mathrm{Hom}(M, N)| < \infty$; (2) $|\mathrm{Ext}^1(M, N)| < \infty$ for any $M, N \in \mathcal{A}$. Some typical examples of finitary categories are the category of finite length modules over some discrete valuation ring R whose residue field R/m is finite, the category of finite dimensional representations of a quiver $\mathcal{Q}$ over a finite field k and the category of coherent sheaves on some projective scheme over a finite field k.

The Hall algebra $\mathcal{H}(\mathcal{A})$ associated to a finitary category $\mathcal{A}$ is an associative algebra, which, as a $\mathbb{Q}$-vector space, has a basis consisting of the isomorphism classes $[X]$ for $X \in \mathcal{A}$ and has the multiplication $[X] * [Y] = \sum_{[L]} \mathrm{g}_{XY}^L [L]$, where $X, Y, L \in \mathcal{A}$ and $\mathrm{g}_{XY}^L = |\{M \subset L | M \simeq X \text{ and } L/M \simeq Y\}|$ is the structure constant related to counting exact sequences $0 \to X \xrightarrow{f} L \xrightarrow{g} Y \to 0$ and is called the *Hall number* (see [5]). Equivalently,

$$\mathrm{g}_{XY}^L = \frac{|\mathcal{M}(X, L)_Y|}{|\mathrm{Aut}X|},$$

where $\mathcal{M}(X, L)_Y$ is the subset of $\mathrm{Hom}(X, L)$ consisting of monomorphisms $f : X \hookrightarrow L$ whose cokernels $\mathrm{Coker}(f)$ are isomorphic to Y. Indeed, g_{XY}^L can be calculated as follows. Define

$$E(X, Y; L) = \{(f, g) \in \mathrm{Hom}(X, L) \times \mathrm{Hom}(L, Y) \mid$$

$$0 \to X \xrightarrow{f} L \xrightarrow{g} Y \to 0 \text{ is an exact sequence}\}.$$

The group $\mathrm{Aut}X \times \mathrm{Aut}Y$ acts freely on $E(X, Y; L)$ and the orbit of $(f, g) \in E(X, Y; L)$ is denoted by $(f, g)^\wedge := \{(af, gc^{-1}) \mid (a, c) \in \mathrm{Aut}X \times \mathrm{Aut}Y\}$. If the orbit space is denoted by $O(X, Y; L) = \{(f, g)^\wedge | (f, g) \in E(X, Y; L)\}$, then

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$g^L_{XY} = |O(X, Y; L)|$. The associativity of the Hall algebra of a finitary category follows from pull–back and push–out constructions. One can refer to [16] and [17] for systematic introductions to this topic.

In the following, we shall describe briefly the current developments of the theory of Hall algebras. The term "Hall Algebra" is due to Ringel, as the generalization of "algebra of partitions" originally constructed in the context of abelian p-groups by Hall [3], which even has a trace back to the problems considered by Steineiz [19]. In the early 90's, Ringel defined the Hall algebra associated to an abelian category in [15]. In particular, when the abelian category $\mathcal{A}$ is the module category of finitely generated modules of a finite dimensional hereditary algebra Λ over a finite field, the positive part of Drinfeld-Jimbo's quantum group $\mathbf{U_q(g)}$ of the Kac–Moody Lie algebra $\mathfrak{g}$ is obtained as the (generic, twisted) Hall algebra of $\mathcal{A}$ if Λ and $\mathfrak{g}$ share the same diagram (see [15, 2]). The theory of Hall algebras is closely related to representation theory and algebraic geometry. Note that due to this link established between the geometry of quiver representations and quantum groups that Lusztig [10] developed his theory of canonical basis. Kashiwara [6] constructed independently such bases (called the crystal basis) by combinatorial methods. In studying a Hall algebra associated to the category of coherent sheaves on the projective line, Kapranov [7] opened a new direction in the theory. One can extend his study to the case of elliptic curves and of surfaces.

A Hall algebra was originally made from an abelian category. After Ringel's discovery, as a generalization, some attempts to strengthen the relationship between the Hall algebra over some abelian category and the quantum group of some Lie algebra have led to the problem of associating some kinds of Hall algebras to categories which are triangulated rather than abelian. One may also naturally ask the following question (see [15]): how to recover the whole Lie algebra and the whole quantum group? A direct way is to use the reduced Drinfeld double to glue together two Borel parts as shown by Xiao [22]. However, this construction is not intrinsic, that is, not naturally induced by the module category of corresponding hereditary algebra Λ. Therefore, one needs to replace the module category by a larger category. It was first pointed out by Xiao [21], cf. also [8], that an extension of the construction of the Hall algebra to the derived category of a finite dimensional hereditary algebra Λ (which is also a triangulated category) might yield the whole quantum group. Unfortunately, as Kapranov pointed out, a direct mimicking of the Hall algebra construction, but with triangles replacing exact sequences, fails to give an associative multiplication. Therefore, we can not use this way to recover the whole quantum groups or enveloping algebras. However, Peng and Xiao [14] defined an analogous multiplication of Hall multiplication over the 2-periodic k-additive triangulated category for some finite field k with the cardinality q. A triangulated category is periodic if the translation functor $T = [1]$ is periodic, i.e., $[1]^d \cong 1$ for some $d \in \mathbb{N}$. Although this multiplication is, in general, not associative, the induced Lie bracket given by the commutator of the nature Hall multiplication satisfies Jacobi identity. This well-defined Hall type algebra provides the realization of Kac-Moody Lie algebras. In [24], we constructed a new multiplication over $\mathbb{Z}/(q-1)$ for the 2-periodic triangulated category which is almost associative for isomorphism classes of indecomposable objects. This refines the construct of Peng and Xiao [14].

On the other hand, Kapranov [8] defined an associative algebra called lattice algebra associated to the derived category of any hereditary category. This associative algebra can be viewed as an analog of the Hall algebra associated to a finitary category. Essentially, to define a Hall algebra is to provide a formula for Hall numbers. However, there is no formula for defining a Hall algebra from any triangulated category. Recently, Toën [20] made a remarkable development in this direction. By using model category and the fiber products of model categories, Toën defined a "derived Hall algebra"–an associative algebra associated to a triangulated category $\mathcal{T}$ with the translation functor [1] which appears as homotopy category of model category whose objects are modules over a sufficiently finitary dg (differential graded) category over a finite field k. He obtained an explicit formula for the structure constant Φ_{XY}^L of an associative multiplication on the rational vector space generated by the isomorphism classes of $\mathcal{T}$. The structure constant Φ_{XY}^L is called the derived Hall number and supplied by a formula of Toën [20, Proposition 5.1] as follows:

$$\Phi_{XY}^L = \frac{|(X, L)_Y|}{|\mathrm{Aut}X|} \cdot \big(\prod_{i>0} \frac{|\mathrm{Hom}(X[i], L)|^{(-1)^i}}{|\mathrm{Hom}(X[i], X)|^{(-1)^i}}\big),$$

where $X, Y, L \in \mathcal{T}$ and $(X, L)_Y$ is the subset of $\mathrm{Hom}(X, L)$ consisting of morphisms $l : X \to L$ whose mapping cones $\mathrm{Cone}(l)$ are isomorphic to Y. Indeed, the structure constant Φ_{XY}^L is related to counting triangles $X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1]$. Note that the Hall algebras defined by Ringel [15] and derived Hall algebras defined by Toën [20] provided the most canonical examples of Hall algebras as a tool of the categorification. To define the derived Hall algebra of any triangulated category under some (homological) finiteness conditions by using Toën formula becomes a very interesting question ([20, Remark 5.3]). In [23], the authors extended derived Hall algebras to Hall algebras associated to triangulated categories $\mathcal{T}$ with some homological finiteness conditions and the Toën formula has the following variant:

$$\Phi_{XY}^L = \prod_{i>0} |\mathrm{Hom}(X[i], Y)|^{(-1)^i} \cdot \sum_{\alpha \in V(X,Y;L)} \frac{|\mathrm{End}(X_1(\alpha))|}{|\mathrm{Aut}(X_1(\alpha))|},$$

where we refer to Proposition 2.1 in Section 2 for the definitions of $V(X, Y; L)$ and $X_1(\alpha)$. In [18], the authors proved that associated to the derived category $\mathcal{D}(\mathcal{A})$ of a hereditary k-category $\mathcal{A}$ over a finite field k, the derived Hall algebra $\mathcal{DH}(\mathcal{A})$ defined by Toën can be identified with the lattice algebra $L(\mathcal{A})$ defined by Kapranov [23] via the "twist and extend" procedure through a suitable subalgebra closely related to the Heisenberg double. However, none of these methods can yet be applied to the periodic triangulated category (in particular, 2-periodic triangulated category (root category)), as it does not satisfy the homological finiteness assumptions.

The purpose of this paper is to define a Hall algebra, associated to a triangulated category $\mathcal{C}$ with the t-periodic functor for any odd number $t > 1$. We prove an analogue of the formula of Toën [20, Proposition 5.1] and for any $X, Y, L \in \mathcal{C}$, the structure constant

$$F_{XY}^L = \frac{|(X, L)_Y|}{|\mathrm{Aut}X|} \cdot \big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}(X[i], L)|^{(-1)^i}}{|\mathrm{Hom}(X[i], X)|^{(-1)^i}}\big)^{\frac{1}{2}}.$$

As an application, we describe the Hall algebra associated to the 3-periodic orbit category of a hereditary abelian category. Note that the t-periodic triangulated

categories do not satisfy the required homological finiteness conditions appeared in [20, 23] and our results can be viewed as non-trivial generalization of Toën's result [20] and Xiao-Xu's result [23]. It will be of interest to deal with those Lie algebras which arise from a Hall algebra over an odd periodic triangulated category.

2. Hall algebras arising from odd-periodic triangulated categories

Let k be a finite field with q elements and $\mathcal{C}$ a k-additive triangulated category with the translation (or shift) functor $T = [1]$ satisfying the following conditions:

(1) the homomorphism space $\text{Hom}(X, Y)$ for any two objects X and Y in $\mathcal{C}$ is a finite dimensional k-space;
(2) the endomorphism ring $\text{End}X$ for any indecomposable object X in $\mathcal{C}$ is a finite dimensional local k-algebra;
(3) $T^t = [1]^t = [t] \cong 1_{\mathcal{C}}$ for some positive integer t.

Then the category $\mathcal{C}$ is called a t-periodic triangulated category and $T = [1]$ is called a t-periodic translation (or shift) functor. Note that the first two conditions imply the validity of the Krull–Schmidt theorem in $\mathcal{C}$, which means that any object in $\mathcal{C}$ can be uniquely decomposed into the direct sum of finitely many indecomposable objects up to isomorphism.

Throughout this paper, we will assume that t is an odd number and $t > 1$. For any X, Y and Z in $\mathcal{C}$, we will use fg to denote the composition of morphisms $f : X \to Y$ and $g : Y \to Z$, and $|S|$ to denote the cardinality of a finite set S.

Given $X, Y; L \in \mathcal{C}$, put

$$W(X, Y; L) = \{(f, g, h) \in \text{Hom}(X, L) \times \text{Hom}(L, Y) \times \text{Hom}(Y, X[1]) \mid$$

$$X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1] \text{ is a triangle}\}.$$

There is a natural action of $\text{Aut}X \times \text{Aut}Y$ on $W(X, Y; L)$. The orbit of $(f, g, h) \in W(X, Y; L)$ is denoted by

$$(f, g, h)^{\wedge} := \{(af, gc^{-1}, ch(a[1])^{-1}) \mid (a, c) \in \text{Aut}X \times \text{Aut}Y\}.$$

The orbit space is denoted by $V(X, Y; L) = \{(f, g, h)^{\wedge} \mid (f, g, h) \in W(X, Y; L)\}$. The radical of $\text{Hom}(X, Y)$ is denoted by $\text{radHom}(X, Y)$ which is the set

$$\{f \in \text{Hom}(X, Y) \mid gfh \text{ is not an isomorphism for any } g : A \to X \text{ and}$$

$$h : Y \to A \text{ with } A \in \mathcal{C} \text{ indecomposable }\}.$$

Denote by $(X, Y)_Z$ the subset of $\text{Hom}(X, Y)$ consisting of the morphisms whose mapping cones are isomorphic to Z.

The Proposition 2.5 in [23] also holds for t–periodic triangulated categories.

Proposition 2.1. For any $Z, L, M \in \mathcal{C}$, we have the following:

(1) any $\alpha = (l, m, n)^{\wedge} \in V(Z, L; M)$ has the representative of the form:

$$Z \xrightarrow{\begin{pmatrix} 0 \\ l_2 \end{pmatrix}} M \xrightarrow{\begin{pmatrix} 0 & m_2 \end{pmatrix}} L \xrightarrow{\begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}} Z[1]$$

where $Z = Z_1(\alpha) \oplus Z_2(\alpha)$, $L = L_1(\alpha) \oplus L_2(\alpha)$, n_{11} is an isomorphism between $L_1(\alpha)$ and $Z_1(\alpha)[1]$ and $n_{22} \in \text{radHom}(L_2(\alpha), Z_2(\alpha)[1])$;

(2)

$$\frac{|(M,L)_{Z[1]}|}{|\mathrm{Aut}L|} = \sum_{\alpha \in V(Z,L;M)} \frac{|\mathrm{End}L_1(\alpha)|}{|n\mathrm{Hom}(Z[1],L)||\mathrm{Aut}L_1(\alpha)|}$$

and

$$\frac{|(Z,M)_L|}{|\mathrm{Aut}Z|} = \sum_{\alpha \in V(Z,L;M)} \frac{|\mathrm{End}Z_1(\alpha)|}{|\mathrm{Hom}(Z[1],L)n||\mathrm{Aut}Z_1(\alpha)|}$$

where $n\mathrm{Hom}(Z[1],L) = \{b \in \mathrm{End}L \mid b = ns$ for some $s \in \mathrm{Hom}(Z[1],L)\}$ and $\mathrm{Hom}(Z[1],L)n = \{d \in \mathrm{End}Z[1] \mid d = sn$ for some $s \in \mathrm{Hom}(Z[1],L)\}$.

Lemma 2.2. *For any $(l,m,n) \in W(Z,L;M)$, we have*

(1) $|n\mathrm{Hom}(Z[1],L)| = \big(\displaystyle\prod_{i=1}^{t} \frac{|\mathrm{Hom}(M[i],L)|^{(-1)^i}}{|\mathrm{Hom}(Z[i],L)|^{(-1)^i}|\mathrm{Hom}(L[i],L)|^{(-1)^i}} \big)^{\frac{1}{2}};$

(2) $|\mathrm{Hom}(Z[1],L)n| = \big(\displaystyle\prod_{i=1}^{t} \frac{|\mathrm{Hom}(Z[i],M)|^{(-1)^i}}{|\mathrm{Hom}(Z[i],L)|^{(-1)^i}|\mathrm{Hom}(Z[i],Z)|^{(-1)^i}} \big)^{\frac{1}{2}}.$

Proof. We only prove the first identity, the proof for the second identity is similar. By applying the functor $\mathrm{Hom}(-,L)$ to the triangle $Z \xrightarrow{l} M \xrightarrow{m} L \xrightarrow{n} Z[1]$, we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(Z[t+1],L) \xrightarrow{n[t]^*} \mathrm{Hom}(L[t],L) \longrightarrow$$

$$\cdots \longrightarrow \mathrm{Hom}(Z[1],L) \xrightarrow{n^*} \mathrm{Hom}(L,L) \longrightarrow \cdots$$

Since $n\mathrm{Hom}(Z[1],L) = $ Image of n^*, $n[t]\mathrm{Hom}(Z[t+1],L) = $ Image of $n[t]^*$ and $[t] \cong 1_{\mathcal{C}}$, the above long exact sequence induces the following long exact sequence

$$0 \longrightarrow n\mathrm{Hom}(Z[1],L) \longrightarrow \cdots \longrightarrow n\mathrm{Hom}(Z[1],L) \longrightarrow 0$$

which implies the desired identity.

$\square$

By Proposition 2.1 and Lemma 2.2, we have the following immediate result.

Corollary 2.3. With the above notations, we have

$$\frac{|(M,L)_{Z[1]}|}{|\mathrm{Aut}L|} \cdot \big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}(M[i],L)|^{(-1)^i}}{|\mathrm{Hom}(L[i],L)|^{(-1)^i}} \big)^{\frac{1}{2}} = \frac{|(Z,M)_L|}{|\mathrm{Aut}Z|} \cdot \big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}(Z[i],M)|^{(-1)^i}}{|\mathrm{Hom}(Z[i],Z)|^{(-1)^i}} \big)^{\frac{1}{2}}.$$

Recall some notations appeared in [23]. Let X, Y, Z, L, L' and M be in $\mathcal{C}$. Define

$$\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]} := \{ \begin{pmatrix} m \\ f \end{pmatrix} \in \mathrm{Hom}(M \oplus X, L) \mid$$

$$\mathrm{Cone}(f) \simeq Y, \mathrm{Cone}(m) \simeq Z[1] \text{ and } \mathrm{Cone}\begin{pmatrix} m \\ f \end{pmatrix} \simeq L'[1]\}$$

and

$$\mathrm{Hom}(L', M \oplus X)_L^{Y,Z[1]} := \{(f', -m') \in \mathrm{Hom}(L', M \oplus X) \mid$$

$$\mathrm{Cone}(f') \simeq Y, \mathrm{Cone}(m') \simeq Z[1] \text{ and } \mathrm{Cone}(f', -m') \simeq L\}.$$

Note that the orbit space of $\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}$ under the action of $\mathrm{Aut}L$ is just the orbit space of $\mathrm{Hom}(L', M \oplus X)_L^{Y,Z[1]}$ under the action of $\mathrm{Aut}L'$ (see [23]), which is denoted by $V(L', L; M \oplus X)_{Y,Z[1]}$.

If in Corollary 2.3, we substitute M by $M \oplus X$ and Z by L', then we can have the following result as an analog of Corollary 2.3 by using a similar proof.

Corollary 2.4. With the above notations, we have

$$\frac{|\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}|}{|\mathrm{Aut}L|} \cdot \Big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}((M \oplus X)[i], L)|^{(-1)^i}}{|\mathrm{Hom}(L[i], L)|^{(-1)^i}}\Big)^{\frac{1}{2}}$$

$$= \frac{|\mathrm{Hom}(L', M \oplus X)_L^{Y,Z[1]}|}{|\mathrm{Aut}L'|} \cdot \Big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}(Z[i], M)|^{(-1)^i}}{|\mathrm{Hom}(Z[i], Z)|^{(-1)^i}}\Big)^{\frac{1}{2}}.$$

For any Z, L and $M \in \mathcal{C}$, set

$$
\begin{aligned}
F_{ZL}^M &:= \frac{|(M, L)_{Z[1]}|}{|\mathrm{Aut}L|} \cdot \Big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}(M[i], L)|^{(-1)^i}}{|\mathrm{Hom}(L[i], L)|^{(-1)^i}}\Big)^{\frac{1}{2}} \\
&= \frac{|(Z, M)_L|}{|\mathrm{Aut}Z|} \cdot \Big(\prod_{i=1}^{t} \frac{|\mathrm{Hom}(Z[i], M)|^{(-1)^i}}{|\mathrm{Hom}(Z[i], Z)|^{(-1)^i}}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

This formula is an analogue of Toën's formula [20, Proposition 5.1]. Set $q = v^2$, then $F_{ZL}^M \in \mathbb{Q}[v, v^{-1}]$. Let $\mathbb{Q}(v, v^{-1})$ be the rational field of $\mathbb{Q}[v, v^{-1}]$. For any $X \in \mathcal{C}$, we denote its isomorphism class by $[X]$.

Now we can define an associative algebra arising from $\mathcal{C}$ using F_{ZL}^M as the structure constant.

Theorem 2.5. *Let $\mathcal{H}(\mathcal{C})$ be the $\mathbb{Q}(v, v^{-1})$-space with the basis $\{u_{[X]} \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by*

$$u_{[X]} * u_{[Y]} = \sum_{[L]} F_{XY}^L u_{[L]},$$

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $u_{[0]}$.

Proof. In order to simplify the notation, for $X, Y \in \mathcal{C}$, we set

$$\{X, Y\} = \prod_{i=1}^{t} |\mathrm{Hom}(X[i], Y)|^{(-1)^i}.$$

For X, Y, Z and $M \in \mathcal{C}$, we will prove that $u_Z * (u_X * u_Y) = (u_Z * u_X) * u_Y$. It is equivalent to prove

$$\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L']} F_{ZX}^{L'} F_{L'Y}^M.$$

We know that

$$\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L]} \frac{|(X, L)_Y|}{|\mathrm{Aut}(X)|\{X, X\}^{\frac{1}{2}}} \{X, L\}^{\frac{1}{2}} \cdot \frac{|(M, L)_{Z[1]}|}{|\mathrm{Aut}(L)|\{L, L\}} \{M, L\}^{\frac{1}{2}}$$

By Proposition 3.5 in [23], $\sum_{[L]} F^L_{XY} F^M_{ZL}$ is equal to

$$\frac{1}{|\mathrm{Aut}X| \cdot \{X,X\}^{\frac{1}{2}}} \cdot \sum_{[L]} \sum_{[L']} \frac{|\mathrm{Hom}(M \oplus X, L)^{Y, Z[1]}_{L'[1]}|}{|\mathrm{Aut}L|} \cdot \frac{\{M \oplus X, L\}^{\frac{1}{2}}}{\{L, L\}^{\frac{1}{2}}}.$$

Dually, $\sum_{[L']} F^{L'}_{ZX} F^M_{L'Y}$ is equal to

$$\frac{1}{|\mathrm{Aut}X| \cdot \{X,X\}^{\frac{1}{2}}} \sum_{[L']} \cdot \sum_{[L]} \frac{|\mathrm{Hom}(L', M \oplus X)^{Y, Z[1]}_{L}|}{|\mathrm{Aut}L'|} \cdot \frac{\{L', M \oplus X\}^{\frac{1}{2}}}{\{L', L'\}^{\frac{1}{2}}}.$$

By Corollary 2.4, the proof of the theorem follows immediately.   □

3. Hall algebras for 3-periodic relative derived categories of hereditary categories

In this section, first of all, we will describe the relative derived categories of t-cycle complexes over hereditary abelian categories, which are the concrete examples of t–periodic triangulated categories. Then, under the case when $t = 3$, we will provide a detailed description of the Hall algebra over a 3–periodic triangulated category.

3.1. t-cycle complexes and relative homotopy categories.
Let $\mathcal{A}$ be an abelian category. Let $t \in \mathbb{N}$ and $t > 1$. Recall from [13] that a t–cycle complex over $\mathcal{A}$ is a complex $\dot{X}_t = (X^i, d^i_X)_{i \in \mathbb{Z}}$ such that $X^i = X^j$ and $d^i_X = d^j_X$ for all $i, j \in \mathbb{Z}$ with $i \equiv j (\mathrm{mod}\, t)$, where all X^i are objects in $\mathcal{A}$ and all $d^i_X : X^i \longrightarrow X^{i+1}$ are morphisms in $\mathcal{A}$ with $d^i_X d^{i+1}_X = 0$. For convenience, we can express $\dot{X}_t$ by using the form:

$$X_1 \xrightarrow{d^1_X} X_2 \xrightarrow{d^2_X} \cdots \longrightarrow X_t \xrightarrow{d^t_X} X_1 .$$

If $\dot{X}_t$ and $\dot{Y}_t$ are two t–cycle complexes, a morphism $\dot{f}_t : \dot{X}_t \longrightarrow \dot{Y}_t$ is a sequence of morphisms $f^i : X^i \longrightarrow Y^i$ in $\mathcal{A}$ such that $f^i = f^j$ for all $i, j \in \mathbb{Z}$ with $i \equiv j (\mathrm{mod}\, t)$ and $d^i_X f^{i+1} = f^i d^i_Y$ for all $i \in \mathbb{Z}$. These morphisms are composed in an obvious way. So all t–cycle complexes with these morphisms constitute an abelian category $\mathcal{C}_t(\mathcal{A})$ which is called the t–cycle complex category.

Let $\dot{f}_t, \dot{g}_t : \dot{X}_t \longrightarrow \dot{Y}_t$ be two morphisms of t–cycle complexes. A relative homotopy $\dot{s}_t$ from $\dot{f}_t$ to $\dot{g}_t$ a sequence of morphisms $s^i : X^i \longrightarrow Y^{i-1}$ in $\mathcal{A}$ such that $s^i = s^j$ for all $i, j \in \mathbb{Z}$ with $i \equiv j (\mathrm{mod}\, t)$ and $f^i - g^i = s^i d^{i-1}_Y + d^i_X s^{i+1}$ for all $i \in \mathbb{Z}$. Morphisms $\dot{f}_t$ and $\dot{g}_t$ are said to be relatively homotopic if there exists a relative homotopy from $\dot{f}_t$ to $\dot{g}_t$. Relative homotopy is an equivalence relation compatible with composition of morphisms. So we can form a new additive category $\mathcal{K}_t(\mathcal{A})$, called the relative homotopy category of t–cycle complexes over $\mathcal{A}$, by considering all t–cycle complexes as objects and the additive group of relative homotopy classes of morphisms from $\dot{X}_t$ to $\dot{Y}_t$ in $\mathcal{C}_t(\mathcal{A})$ as the group of morphisms from $\dot{X}_t$ to $\dot{Y}_t$ in $\mathcal{K}_t(\mathcal{A})$. As in usual complex categories, one can define quasi-isomorphisms in $\mathcal{C}_t(\mathcal{A})$ and $\mathcal{K}_t(\mathcal{A})$. Localizing $\mathcal{K}_t(\mathcal{A})$ with respect to the set of all quasi-isomorphisms, one can get an additive category, denoted by $\mathcal{D}_t(\mathcal{A})$, called the relative derived category of t–cycle complexes over $\mathcal{A}$ (see [9, Section 3.2]). As in the appendix of [13], $\mathcal{K}_t(\mathcal{A})$ is a triangulated category with the translation functor [1] which is the stable category of a Frobenius category. In the same way, $\mathcal{D}_t(\mathcal{A})$ is also a triangulated

category with the translation functor $[1]$. As usual we denote by $\mathcal{C}^b(\mathcal{A})$ the category of bounded complexes over $\mathcal{A}$, and by $\mathcal{K}^b(\mathcal{A})$ the homotopy category of bounded complexes over $\mathcal{A}$. Let $\mathcal{D}^b(\mathcal{A})$ be the derived category of $\mathcal{A}$ obtained from $\mathcal{K}^b(\mathcal{A})$ by localizing with respect to the set of quasi–isomorphisms. Then $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$ are all triangulated categories with the translation functor–the shift functor $[1]$.

As in $[13]$, we can define a functor $F : \mathcal{C}^b(\mathcal{A}) \longrightarrow \mathcal{C}_t(\mathcal{A})$ as follows. For $\dot{X} = (X^i, d_X^i) \in \mathcal{C}^b(\mathcal{A})$, set $F\dot{X} = ((F\dot{X})^i, d_{F\dot{X}}^i)$ where $F\dot{X} = \oplus_{m \in \mathbb{Z}} X^{i+mt}$ and $d_{F\dot{X}}^i = (d_{sm}^i)_{s,m \in \mathbb{Z}}$ such that $d_{sm}^i : X^{i+st} \longrightarrow X^{(i+1)+mt}$ with $d_{sm}^i = 0$ for $s \neq m$ and $d_{ss}^i = d_X^i$ for all $s \in \mathbb{Z}$. For $\dot{f} = (f^i) : \dot{X} \longrightarrow \dot{Y}$ in $\mathcal{C}^b(\mathcal{A})$, set $F(\dot{f}) = (g^i)_{i \in \mathbb{Z}} : F\dot{X} \longrightarrow F\dot{Y}$ where $g^i = (g_{sm}^i)_{s,m \in \mathbb{Z}}$ such that $g_{sm}^i : X^{i+st} \longrightarrow Y^{i+mt}$ with $g_{sm}^i = 0$ for $s \neq m$ and $g_{ss}^i = f^i$ for all $s \in \mathbb{Z}$. It is not difficult to check that F is a well–defined functor.

It is not hard to verify that F is a Galois covering with the Galois group $G = \langle [1]^t \rangle$ and F is exact, i.e. F sends a chainwise split exact sequence to a chainwise split exact sequence. In $[13]$, the authors proved that F induces a Galois covering from $\mathcal{K}^b(\mathcal{A})$ to $\mathcal{K}_t(\mathcal{A})$ with the Galois group $\langle [1]^t \rangle$ and it is exact, i.e. it sends a triangle to a triangle. In the same way, F induces a Galois covering from $\mathcal{D}^b(\mathcal{A})$ to $\mathcal{D}_t(\mathcal{A})$, denoted still by F. For any $X, Y \in \mathcal{A}$, let $\tilde{X} := F(X)$ and $\tilde{Y} := F(Y)$, we have

$$(3.1) \qquad \mathrm{Hom}_{\mathcal{D}_t(\mathcal{A})}(\tilde{X}, \tilde{Y}) \cong \bigoplus_{F(X')=\tilde{X}} \mathrm{Hom}(X', Y) \cong \bigoplus_{F(Y')=\tilde{Y}} \mathrm{Hom}(X, Y').$$

The following proposition is similar to Proposition 3.2 in $[9]$.

Proposition 3.1. Let $\mathcal{A}$ be a hereditary (abelian) category, i.e., $\mathrm{Ext}^i(-,-) = 0$ for $i > 1$ and $F : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}_t(\mathcal{A})$ be the Galois covering functor for $t \in \mathbb{N}$ and $t > 1$. Then F is dense.

Proof. Given a t-cycle complex $\dot{X}$ with the form as follows:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \longrightarrow X_t \xrightarrow{f_t} X_1 .$$

For $f_1 : X_1 \to X_2$, there exists $L \in \mathcal{A}$ such that the following diagram

is commutative.

This diagram induces a short exact sequence in $\mathcal{C}_t(\mathcal{A})$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ker}f_1 & == & \mathrm{Ker}f_1 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & \mathrm{Ker}f_1 \\
\end{array}
$$

Note that since $\mathrm{Im}f_t \subseteq \mathrm{Ker}f_1$, we can write $f_t : X_t \to \mathrm{Ker}f_1$ without causing any confusion. Since the t–cycle complex on line 2 is contractible, two t–cycle complexes on line 3 and 4 are quasi-isomorphic to each other. Let $X^\bullet \in \mathcal{D}^b(\mathcal{A})$ be the complex
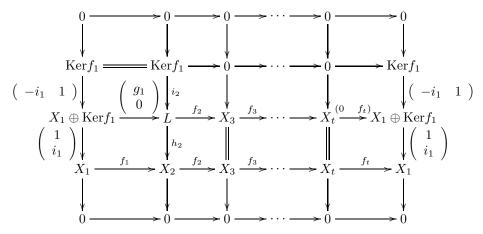
$$\cdots \longrightarrow 0 \longrightarrow X_1 \xrightarrow{f_1} \cdots \longrightarrow X_t \xrightarrow{f_t} \mathrm{Ker}f_1 \longrightarrow 0 \longrightarrow \cdots .$$

Then $F(X^\bullet)$ is just the t–cycle complex on line 3 and then isomorphic to $\dot{X}$ in $\mathcal{D}_t(\mathcal{A})$. Hence, F is dense. $\qquad\square$

Remark 3.2. Now let Λ be a basic finite dimensional associative algebra with unity over a field. Let $\mathrm{mod}\,\Lambda$ be the category of all finitely generated (right) Λ–modules and $\mathcal{P}$ the full subcategory of $\mathrm{mod}\,\Lambda$ with projective Λ-modules as objects. For simplicity, we write $\mathcal{D}^b(\Lambda)$ instead of $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ and call it the derived category of Λ. Let $\mathcal{K}_t(\mathcal{P})$ be the relative homotopy category of t-cycle complex over $\mathcal{P}$. If Λ is hereditary, then $\mathcal{D}^b(\Lambda)$ is a Galois cover of $\mathcal{K}_t(\mathcal{P})$ with the Galois group $\langle [1]^t \rangle$ and the covering functor is exact and dense, where $[1]$ is the translation functor of the triangulated category $\mathcal{D}^b(\Lambda)$. Under the covering functor $F' : \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{K}_t(\mathcal{P})$, $\mathrm{mod}\,\Lambda$ can be fully embedded in $\mathcal{K}_t(\mathcal{P})$. This full embedding may be obtained directly by sending Λ–modules to the t–cycle complexes over $\mathcal{P}$ which is naturally induced by their projective resolutions. Similarly with Happel's discussions in [4], one can prove that $\mathcal{K}_t(\mathcal{P})$ has Auslander–Reiten triangles. Specially in case $t = 2$, the orbit category $\mathcal{D}^b(\Lambda)/[1]^2$ is called the root category of Λ by Happel. Then $\mathcal{D}^b(\Lambda)/[1]^2 \simeq \mathcal{K}_2(\mathcal{P})$ is a triangulated category and the covering $F' : \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(\Lambda)/[1]^2$ is exact. Therefore, $\mathcal{D}^b(\Lambda)/[1]^2 \simeq \mathcal{D}_2(\mathrm{mod}\,\Lambda)$.

There is a full embedding of $\mathcal{A}$ into $\mathcal{D}_t(\mathcal{A})$ which sends each object X of $\mathcal{A}$ into the stalk complex $0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} X \xrightarrow{0} 0$ be a t–cycle complex. We will identify this complex with X.

Let $\mathrm{ind}\mathcal{A}$ be the set of isomorphism classes of indecomposable objects in $\mathcal{A}$. Proposition 3.1 implies that if $\mathcal{A}$ is hereditary, then the set of isomorphism classes of indecomposable objects in $\mathcal{D}_t(\mathcal{A})$ is

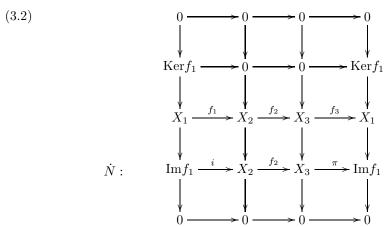$$\{X[i] \mid i = 0, 1, \cdots, t - 1, X \in \mathrm{ind}\mathcal{A}\}.$$

In the following, we will only deal with the case when $t = 3$ for simplicity.

Proposition 3.3. Let $\dot{X} : X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_1$ be a 3–cycle complex over a hereditary abelian category $\mathcal{A}$. Then in $\mathcal{D}_3(\mathcal{A})$, $\dot{X}$ is isomorphic to
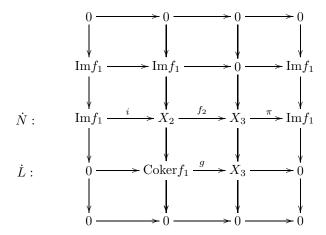
$$\mathrm{Ker}f_1 \xrightarrow{\;0\;} \mathrm{Ker}f_2/\mathrm{Im}f_1 \xrightarrow{\;0\;} \mathrm{Coker}f_2 \xrightarrow{\;f\;} \mathrm{Ker}f_1$$

where f is naturally induced by f_3.

Proof. In $\mathcal{C}_3(\mathcal{A})$, there exists a natural short exact sequence involving $\dot{X}$ as follows:

(3.2)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ker}f_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Ker}f_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Im}f_1 & \xrightarrow{i} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{\pi} & \mathrm{Im}f_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

($\dot{N}$ labels the fourth row.)

where i is the natural embedding and π is induced by f_3. The complex $\dot{N}$ in above diagram induces the following short exact sequence in $\mathcal{C}_3(\mathcal{A})$:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Im}f_1 & \longrightarrow & \mathrm{Im}f_1 & \longrightarrow & 0 & \longrightarrow & \mathrm{Im}f_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Im}f_1 & \xrightarrow{i} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{\pi} & \mathrm{Im}f_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Coker}f_1 & \xrightarrow{g} & X_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

($\dot{N}$ labels the third row, $\dot{L}$ labels the fourth row.)

where g is induced by f_2. Hence, $\dot{N}$ is isomorphic to the bottom complex (denoted by $\dot{L}$) in $\mathcal{D}_3(\mathcal{A})$. Since the complex $0 \longrightarrow \mathrm{Im}f_2 \longrightarrow X_3 \longrightarrow 0$ is isomorphic to $\mathrm{Coker}f_2[1]$ in $\mathcal{D}_3(\mathcal{A})$, we obtain a triangle in $\mathcal{D}_3(\mathcal{A})$

$$\mathrm{Ker}f_2/\mathrm{Im}f_1[2] \longrightarrow \dot{L} \longrightarrow \mathrm{Coker}f_2[1] \xrightarrow{h} \mathrm{Ker}f_2/\mathrm{Im}f_1 \;.$$

By using isomorphisms in (3.1), we have

$$\mathrm{Hom}_{\mathcal{D}_3(\mathcal{A})}(\mathrm{Coker}f_2[1], \mathrm{Ker}f_2/\mathrm{Im}f_1) = 0.$$

Hence, $h = 0$ and then the triangle is split. So $\dot{N}$ is isomorphic to $\mathrm{Ker} f_2/\mathrm{Im} f_1[2] \oplus \mathrm{Coker} f_2[1]$. By using the isomorphisms in (3.1) again, we deduce that the short exact sequence (3.2) induces the triangle

$$\mathrm{Ker} f_1 \longrightarrow \dot{X} \longrightarrow \mathrm{Ker} f_2/\mathrm{Im} f_1[2] \oplus \mathrm{Coker} f_2[1] \overset{t}{\longrightarrow} \mathrm{Ker} f_1[1]$$

where $t = \begin{pmatrix} 0 \\ f[1] \end{pmatrix}$ and f is induced by f_3. The proposition is proved. $\qquad\square$

By using Proposition 3.3, we can easily decompose any object in $\mathcal{D}_3(\mathcal{A})$ into the direct sum of indecomposable objects. Indeed, for the given 3-cycle complex $\dot{X} : X_1 \overset{f_1}{\longrightarrow} X_2 \overset{f_2}{\longrightarrow} X_3 \overset{f_3}{\longrightarrow} X_1$, if $X_2 = 0$, then in $\mathcal{D}_3(\mathcal{A})$, $\dot{X}$ is isomorphic to $0 \longrightarrow \mathrm{Ker} f_2 \overset{0}{\longrightarrow} \mathrm{Coker} f_2 \longrightarrow 0$. This implies that in $\mathcal{D}_3(\mathcal{A})$, the complex $\dot{X}$ in Proposition 3.3 is isomorphic to

$$\mathrm{Ker} f_2/\mathrm{Im} f_1[2] \oplus \mathrm{Ker} f[1] \oplus \mathrm{Coker} f.$$

Now let $\mathcal{A}$ be the category of all finitely generated modules of a basic finite dimensional associative hereditary algebra over a finite field k. Let U be the associative and unital $\mathbb{Q}(v, v^{-1})$-algebra generated by the set

$$\{u^{[i]}_{[X]} \mid i = 0, 1, 2, X \in \mathcal{A}\}$$

with the following generating relations:

(1) for $n = 0, 1, 2$, $u^{[n]}_{[X]} \cdot u^{[n]}_{[Y]} = \sum_{[L]} F^L_{XY} \cdot u^{[n]}_{[L]}$;

(2) for $n = 0, 1$, $u^{[n]}_{[X]} \cdot u^{[n+1]}_{[Y]} = \sum_{[K],[C]} F^{K[1]\oplus C}_{X, Y[1]} \cdot u^{[n+1]}_{[K]} \cdot u^{[n]}_{[C]}$;

(3) $u^{[2]}_{[X]} \cdot u^{[0]}_{[Y]} = \sum_{[K],[C]} F^{K[1]\oplus C}_{X, Y[1]} \cdot u^{[0]}_{[K]} \cdot u^{[2]}_{[C]}$.

Theorem 3.4. *Let $\mathcal{H}(\mathcal{D}_3(\mathcal{A}))$ be the $\mathbb{Q}(v, v^{-1})$-algebra defined in Theorem 2.5. Then there is an isomorphism of algebras from U to $\mathcal{H}(\mathcal{D}_3(\mathcal{A}))$.*

Proof. There is a natural linear map of $\mathbb{Q}(v, v^{-1})$-spaces

$$\Phi : U \to \mathcal{H}(\mathcal{D}_3(\mathcal{A}))$$

defined by $\Phi(u^{[i]}_{[X]}) = u_{[X[i]]}$. It is clear that the relations $(1), (2)$ and (3) in the Definition of U are satisfied in $\mathcal{H}(\mathcal{D}_3(\mathcal{A}))$. Hence, Φ is a homomorphism of algebras. Now we define a partial order over the set of isomorphism classes of objects in $\mathcal{D}_3(\mathcal{A})$. Any object in $\mathcal{D}_3(\mathcal{A})$ is isomorphic to $X \oplus Y[1] \oplus Z[2]$ for some objects X, Y and Z in $\mathcal{A}$. We say

$$[X_1 \oplus Y_1[1] \oplus Z_1[2]] < [X_2 \oplus Y_2[1] \oplus Z_2[2]]$$

if $\underline{\dim} X_1 \leq \underline{\dim} X_2, Y_1 \leq \underline{\dim} Y_2$ and $\underline{\dim} X_3 \leq \underline{\dim} Y_3$ and at least one is not the equation. We claim that Φ is surjective. For any X, Y and Z in $\mathcal{A}$, we have

$$(3.3) \quad u_{[X\oplus Y[1]\oplus Z[2]]} = u_{Z[2]} \cdot u_{Y[1]} \cdot u_X - \sum_{[K],[C]; K \neq X, C \neq Z} F^{K[1]\oplus C}_{Z, X[1]} u_{[K\oplus Y[1]\oplus C[2]]}.$$

It is clear that $[K \oplus Y[1] \oplus C[2]] < [X \oplus Y[1] \oplus Z[2]]$ for any K, C occurred in the right hand of the above equation since $K \neq Z$ and $C \neq Y$ imply that $\underline{\dim}K < \underline{\dim}Z$ and $\underline{\dim}C < \underline{\dim}Y$. We use the induction on the partial order and deduce that there exists $u \in U$ such that $\Phi(u) = u_{[X \oplus Y[1] \oplus Z[2]]}$. Hence, Φ is surjective. Next, we prove that Φ is injective. For any $\mu \in U$, it can be reformulated to have the following form

$$(3.4) \qquad \mu = \sum_{[X],[Y],[Z]} a_{XYZ} \cdot u_{[Z]}^{[2]} \cdot u_{[Y]}^{[1]} \cdot u_{[X]}^{[0]}.$$

where $a_{XYZ} \in \mathbb{Q}(v, v^{-1})$. Indeed, it is enough to check the case when μ is equal to $u_{[X]}^{[0]} \cdot u_{[Z]}^{[2]}$, $u_{[X]}^{0} \cdot u_{[Y]}^{[1]}$ or $u_{[Y]}^{[1]} \cdot u_{[Z]}^{[2]}$. By following the relation (3) in the Definition of U, we have

$$u_{[X]}^{[0]} \cdot u_{[Z]}^{[2]} = u_{[Z]}^{[2]} \cdot u_{[X]}^{[0]} - \sum_{[K],[C];[K \oplus C[2]] < [X \oplus Z[2]]} u_{[K]}^{[0]} \cdot u_{[C]}^{[2]}.$$

By the induction on the partial order, we can rewrite $u_{[X]}^{[0]} \cdot u_{[Z]}^{[2]}$ with the form (3.4). The discussions of $u_{[X]}^{0} \cdot u_{[Y]}^{[1]}$ and $u_{[Y]}^{[1]} \cdot u_{[Z]}^{[2]}$ are similar. By using the expression (3.4), we assume that

$$\Phi(\mu) = \sum_{[X],[Y],[Z]} a_{XYZ} \cdot u_{[Z[2]]} \cdot u_{[Y[1]]} \cdot u_{[X]} = 0.$$

However, by using the expression (3.3), we have

$$\begin{aligned} &u_{[Z[2]]} \cdot u_{[Y[1]]} \cdot u_{[X]} \\ =\ &u_{[X \oplus Y[1] \oplus Z[2]]} + \sum_{[K],[C];[K \oplus Y[1] \oplus C[2]] < [X \oplus Y[1] \oplus Z[2]]} F_{Z,X[1]}^{K[1] \oplus C} u_{[K \oplus Y[1] \oplus C[2]]}. \end{aligned}$$

Hence, we obtain

$$\sum_{[X],[Y],[Z]} a_{XYZ} \cdot u_{[X \oplus Y[1] \oplus Z[2]]} = 0.$$

This implies $a_{XYZ} = 0$ and then $\mu = 0$. The proof of the theorem is completed. $\square$

Remark 3.5. Since $X = X[3]$ for any $X \in \mathcal{D}_3(\mathcal{A})$, the Grothendieck group of $\mathcal{D}_3(\mathcal{A})$ is $\mathbb{Z}_2$-graded. It will be of interest to deal with the $\mathbb{Z}_2$-graded Lie algebras which arise from a Hall algebra over a 3-periodic triangulated category.

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